The Fundamental Theorem of Calculus:

Two major components of Calculus are

① Differentiations — study of tangents

② Integrals — study of area under curves.

Q:- Is there any way to relate these two components?

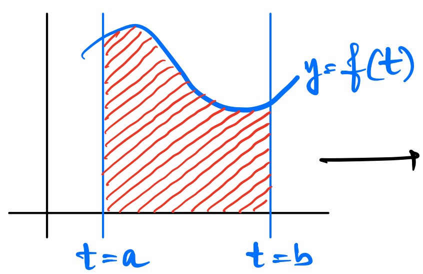
Ans:- YES

Let's understand how!!

We start with

$$\int_a^b f(t)\,dt =$$

a continuous function

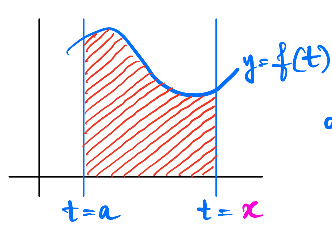


⟶ gives a number

where a, b are fixed.

Now we keep a fixed & choose b to be 'x', a variable.
Then it's like end t=a is fixed & t=b is a slider.

$$\int_a^x f(t)\,dt =$$



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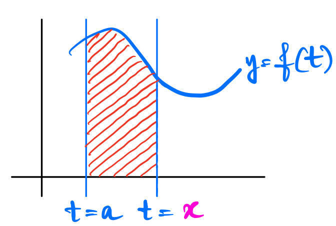


in this case, it's not just a number, it's an area function, depending on x.

So, let

$$F(x)=\int_a^x f(t)\,dt \quad \rightsquigarrow \text{ Accumulation function.}$$

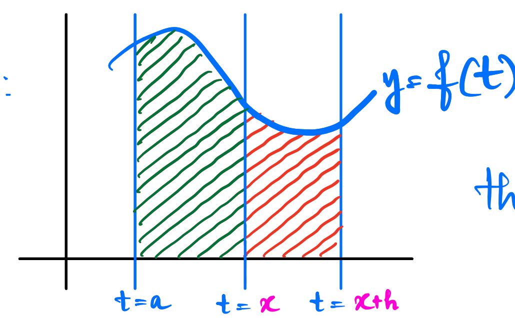
Now what happens if we try computing $F'(x)$?

By definition, $F'(x)=\lim_{h\to 0}\dfrac{F(x+h)-F(x)}{h}$

Here, $F(x+h)=\int_a^{x+h} f(t)\,dt$

Now, $F(x+h)-F(x)=\int_a^{x+h} f(t)\,dt-\int_a^{x} f(t)\,dt$

geometrically:

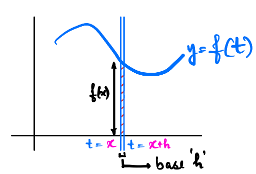


the area in red.

$$=\int_x^{x+h} f(t)\,dt.$$

So, $$\frac{F(x+h)-F(x)}{h}=\frac{\int_x^{x+h} f(t)\,dt}{h}$$

Now, when $h\to 0$ we get



$$\therefore \underbrace{\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}}_{=F'(x)}=\lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}=\lim_{h\to 0}\frac{\text{Area of a rectangle of base } h \text{ \& height } f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)\cdot h}{h}=\lim_{h\to 0} f(x)=f(x)$$

Formal Statement: Suppose f is a continuous function on $[a,b]$.

Let $F(x) = \int_a^x f(t)\,dt$. Then $F'(x) = f(x)$

or

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

In other words: Every continuous function f possesses an anti-derivative, namely

$$F(x) = \int_a^x f(t)\,dt$$

Another form of the Fundamental Theorem of Calculus (FTC):

$$\int_a^b f(t)\,dt = F(b) - F(a),$$

where, $F(x) = \int_a^x f(t)\,dt$ is known.

Example:- ① Find the derivative of $\int_0^x \sqrt{t^2+1}\,dt$

$$\frac{d}{dx}\int_0^x \sqrt{t^2+1}\,dt = \sqrt{x^2+1}$$

② For $f(x) = x^2$, $F(x) = \frac{x^3}{3}$ is known.

$$\int_0^2 x^2\,dx = \frac{x^3}{3}\Big|_0^2 = \underset{F(a)}{\underset{\parallel}{\left(\frac{2^3}{3}\right)}} - \underset{F(b)}{\underset{\parallel}{\left(\frac{0^3}{3}\right)}}$$

$$= \frac{8}{3} - 0 = \frac{8}{3}$$